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Evolution of twofold degenerate two-level system. Geometrical effects

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Abstract. Geometrical effects occurring in the cyclic evolution of twofold degenerate two-level system are considered. A projector representation for the objects involved (Hamiltonian, evolution operator) is constructed. The explicit expressions for connections and curvatures are quoted. Particular cases are considered and found to be in full correspondence with previously obtained results.

1. Introduction

It is well known [1] that during the cyclic adiabatic evolution multi-level (quantum) system acquires the geometric phase factor, or Berry's phase. Simon has shown [2] that it is precisely the holonomy in a Hermitian line bundle since the adiabatic theorem naturally defines a connection in such a bundle. In the case of degenerated systems this factor is essentially non-Abelian [3]. Geometric properties of Berry's phase for multi-level systems has been widely studied (see, for instance, [4]) and it has been found [5] that its calculation often reduces to the explicit calculation of Riemannian connections in bundles over complex Grassmannian manifolds. The general group-theoretical approach to the two-level system with m-fold and n-fold degenerate respective levels has been applied in [6]. However, the common expressions for the connection and geometric phase should be adapted for certain physical problems and the question about the appropriate parametrization arises.

In this paper we concentrate on the investigation of the two-level system with twofold degeneracy of each level. The specific character of the problem allows us to introduce a quaternionic representation for the physical objects involved (evolution operator, Hamiltonian). Not only does it simplify the calculations, but it also permits us to extract easily some particular cases.

The first one corresponds to the special form of the two-level twofold degenerate Hamiltonian which: (a) is employed to manage the adiabatic evolution of a fermion system with time-reversal invariance, e.g. two Kramers doublets [7]; (b) is equivalent to the Hamiltonian of a system with quadrupole interaction [8, 9]. Mathematical aspects of similar Hamiltonians have been discussed in [10].

The second particular case is the system with one twice-degenerate level and another non-degenerate one. Such a model has been considered recently [11–13], and here we recover the already known results.

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It is relevant to note that we shall use the technique of projector-valued operators which appears to be one of the most powerful tools in gauge-field theory. This will allow us to obtain straightforward expressions for curvatures as well as for connections.

2. Evolution of a twofold degenerate two-level system

2.1. The general form of the Hamiltonian

The Hamiltonian of the twofold degenerate two-level system at arbitrary time is given by

$$H(t) = U(t)\,H(0)\,U^{\dagger}(t) \tag{1}$$

where

$$H(0) = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \tag{2}$$

Without loss of generality one can think of the unitary evolution operator $U = \exp(\mathrm{i}M)$ as taking values in the coset space $U(4)/(U(2) \times U(2)) = \mathbb{CG}(4,2)$, i.e. the complex Grassmannian manifold. Then it can be represented due to

$$M = \begin{pmatrix} 0 & B \\ B^{\dagger} & 0 \end{pmatrix} \tag{3}$$

where $B \in \mathfrak{gl}(2,\mathbb{C})$. Because of the isomorphism between $\mathfrak{gl}(2,\mathbb{C})$ and the complex quaternions algebra, one can put $B = u + \mathrm{i}v$, $u = u_4 e_0 + u_k e_k$ and $v = v_4 e_0 + v_k e_k$ being real quaternions.

We recall that the quaternions algebra is defined by the relations

$$e_k e_l = \varepsilon_{klm} e_m - \delta_{kl} e_0 \qquad e_0 e_k = e_k e_0 \qquad e_0^2 = e_0 \qquad (k, l, m = 1, 2, 3). \tag{4}$$

The operation of quaternion conjugation $\bar{u} = u_4 e_0 - u_k e_k$ is an antiautomorphism ($\overline{uv} = \bar{v}\,\bar{u}$) and it enables one to introduce the norm $|u| = \sqrt{u\bar{u}}$, scalar $u_{\mathrm{S}} = \frac{1}{2}(u + \bar{u})$ and vector $u_{\mathrm{V}} = \frac{1}{2}(u - \bar{u})$ parts of the quaternion. Further, we shall use a σ-matrix representation of the basis (4):

$$e_0 = 1_2 \qquad e_k = -\mathrm{i}\sigma_k. \tag{5}$$

Then,

$$M(t) = \gamma_a u_a(t) + \mathrm{i}\gamma_5 \gamma_a v_a(t) \qquad (a = 1, \ldots, 4) \tag{6}$$

with the set of γ-matrices chosen as

$$\gamma_k = \begin{pmatrix} 0 & -\mathrm{i}\sigma_k \\ \mathrm{i}\sigma_k & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Notice that $H(0) = \gamma_5$.

The minimal equation for M $\left(M^2\right)$ is

$$M^4 - 2RM^2 + \left(R^2 - |\boldsymbol{r}|^2\right) = 0 \tag{8}$$

where $R = |u|^2 + |v|^2$ and $\boldsymbol{r} = 2(v\bar{u})_{\mathrm{V}}$. Making use of it, one can calculate

$$U(t) = P_+ \cos\sqrt{\lambda_+} + P_- \cos\sqrt{\lambda_-} + \mathrm{i}M\left(P_+ \frac{\sin\sqrt{\lambda_+}}{\sqrt{\lambda_+}} + P_- \frac{\sin\sqrt{\lambda_-}}{\sqrt{\lambda_-}}\right) \tag{9}$$

where $\lambda_\pm = R \pm |\boldsymbol{r}|$ are the characteristic roots of M^2 and $P_\pm = \frac{1}{2}(1 \pm Q)$ are the projective operators: $P_\pm^2 = P_\pm$, $(P_\pm)^\dagger = P_\pm$, $P_+ P_- = P_- P_+ = 0$;

$$Q = \mathrm{i}\begin{pmatrix} \boldsymbol{n} & 0 \\ 0 & \boldsymbol{k} \end{pmatrix}$$

$$\boldsymbol{n} = \frac{(v\bar{u})_{\mathrm{V}}}{|(v\bar{u})_{\mathrm{V}}|} \qquad \boldsymbol{k} = \frac{(\bar{u}v)_{\mathrm{V}}}{|(\bar{u}v)_{\mathrm{V}}|}.$$

Besides this, $P_\pm$ commute with M.

Taking these definitions into account, it is easy to deduce that

$$M^2 = P_+\lambda_+ + P_-\lambda_-. \tag{10}$$

We duplicate the above expressions in more usual terms

$$M^2 = R + \mathrm{i}\gamma_5 \Sigma_{ab} 2 v_a u_b \tag{11}$$

$$|\boldsymbol{r}| Q = \gamma_5 \Sigma_{ab} 2 v_a u_b \tag{12}$$

$$\Sigma_{ab} = \tfrac{1}{2}[\gamma_a, \gamma_b] \tag{13}$$

$$|\boldsymbol{r}| = 2\sqrt{|v|^2|u|^2 - (v_a u_a)^2}. \tag{14}$$

Verifying the identity

$$U H(0) = H(0)\, U^\dagger \tag{15}$$

we find out that

$$H(t) = U^2(t)\, H(0) = \exp(2\mathrm{i}M) H(0) \tag{16}$$

where

$$U^2 = P_+ \cos 2\sqrt{\lambda_+} + P_- \cos 2\sqrt{\lambda_-} + 2iM\left(P_+ \frac{\sin 2\sqrt{\lambda_+}}{2\sqrt{\lambda_+}} + P_- \frac{\sin 2\sqrt{\lambda_-}}{2\sqrt{\lambda_-}}\right). \tag{17}$$

Due to the remarkable property

$$\mathrm{i}P_\pm M H(0) = P_\pm \begin{pmatrix} 0 & -a_\pm \\ \bar{a}_\pm & 0 \end{pmatrix} \tag{18}$$

with $a_\pm = v \pm \boldsymbol{n}u$ being real quaternions and $|a_\pm|^2 = \lambda_\pm$, the Hamiltonian (1) takes the form

$$\begin{aligned} H(t) = P_+ &\begin{pmatrix} \cos 2|a_+| & \dfrac{\sin 2|a_+|}{|a_+|} a_+ \\ \dfrac{\sin 2|a_+|}{|a_+|} \bar{a}_+ & -\cos 2|a_+| \end{pmatrix} + P_- \begin{pmatrix} \cos 2|a_-| & \dfrac{\sin 2|a_-|}{|a_-|} a_- \\ \dfrac{\sin 2|a_-|}{|a_-|} \bar{a}_- & -\cos 2|a_-| \end{pmatrix} \\ &= P_+(\cos 2|a_+|\,\gamma_5 + \sin 2|a_+|\,\gamma_a \hat{x}_a) + P_-(\cos 2|a_-|\gamma_5 + \sin 2|a_-|\,\gamma_a \hat{y}_a) \\ &\quad 0 \leqslant |a_+|, |a_-| \leqslant \tfrac{1}{2}\pi. \end{aligned} \tag{19}$$

Defining

$$\tan|a_+| = |x| \qquad \tan|a_-| = |y| \tag{20}$$

$$\frac{a_+}{|a_+|} = \frac{x}{|x|} \equiv \hat{x} \qquad \frac{a_-}{|a_-|} = \frac{y}{|y|} \equiv \hat{y} \tag{21}$$

we obtain

$$H = P_+ H_0(x) + P_- H_0(y) \tag{22}$$

where

$$H_0(x) = \frac{1}{1 + |x|^2}\begin{pmatrix} 1 - |x|^2 & 2x \\ 2\bar{x} & -\left(1 - |x|^2\right) \end{pmatrix} \tag{23}$$

and $P_\pm$ are expressed through $\boldsymbol{n}$ and $\boldsymbol{k}$ which have in the new notation the following form:

$$\boldsymbol{n} = \frac{(x\bar{y})_{\mathrm{V}}}{|(x\bar{y})_{\mathrm{V}}|} \qquad \boldsymbol{k} = \frac{(\bar{y}x)_{\mathrm{V}}}{|(\bar{y}x)_{\mathrm{V}}|}. \tag{24}$$

It is relevant to note that $P_\pm$ commute with both $H_0(x)$ and $H_0(y)$.

Expression (22) defines the most general form of the Hamiltonian which manages the dynamics of twofold degenerate two-level system. It depends on eight parameters—x_a and y_a—which are assumed to vary slowly in time. The eigenstate problem posed in the adiabatic approximation

$$H(t)|\Psi(t)\rangle = E_n(t)|\Psi(t)\rangle \tag{25}$$

can be readily solved: the columns of the evolution operator are these eigenstates

$$U = P_+U_0(x) + P_-U_0(y) \tag{26}$$

$$U_0(x) = \frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} 1 & -x \\ \bar{x} & 1 \end{pmatrix}$$
$$= \cos|a_+| + \sin|a_+|\gamma_a\gamma_5\hat{x}_a. \tag{27}$$

The eigenvalues are $E_\pm(t) = \text{constant} = \pm 1$, +1, corresponding to the first two columns and -1 to the second ones.

2.2. Connections and Berry's phases

If we consider the problem in the adiabatic limit, then, according to [1–3], during the cyclic evolution a system acquires a phase factor which consists of two parts: dynamical and geometrical. The latter is an element of the holonomy group and depends on the path in parameters space only:

$$V[C] = \mathcal{P}\exp\left(-\oint_C A_\alpha(r(\tau))\frac{\mathrm{d}r^\alpha}{\mathrm{d}\tau}\,\mathrm{d}\tau\right). \tag{28}$$

The connection A_α is determined with help of eigenstates $\{|n_i\rangle\}$ forming the linear subspace corresponding to the same eigenvalue:

$$(A_\alpha)_{ij} = \langle n_i|\frac{\partial}{\partial r^\alpha}|n_j\rangle. \tag{29}$$

In our case it is natural to define

$$A = \left(U^\dagger\,\mathrm{d}U\right)_{11}$$

where subscripts denote the corresponding 2×2-matrix blocks (quaternions in the σ-matrix representation). So A is a 1-form taking values in the Lie algebra $\mathfrak{u}(2)$. It is expected to decompose into $\mathfrak{u}(2) = \mathfrak{su}(2)\oplus\mathfrak{u}(1)$ parts:

$$A = \boldsymbol{A} - \mathrm{i}A_0. \tag{30}$$

The calculations yield

$$\boldsymbol{A} = \boldsymbol{n}\left(\boldsymbol{n}\boldsymbol{Z}^{(+)}\right) + (\boldsymbol{W} - \boldsymbol{n}(\boldsymbol{n}\boldsymbol{W})) + c[\boldsymbol{n}\,\mathrm{d}\boldsymbol{n}] \tag{31}$$

$$A_0 = \left(\boldsymbol{n}\boldsymbol{Z}^{(-)}\right) \tag{32}$$

where

$$\boldsymbol{Z}^{(\pm)} = \frac{1}{2}\left(\frac{(x\,\mathrm{d}\bar{x})_{\mathrm{V}}}{1+|x|^2} \pm \frac{(y\,\mathrm{d}\bar{y})_{\mathrm{V}}}{1+|y|^2}\right) \tag{33}$$

$$\boldsymbol{W} = \frac{1}{2}\frac{(y\,\mathrm{d}\bar{x})_{\mathrm{V}} + (x\,\mathrm{d}\bar{y})_{\mathrm{V}}}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \tag{34}$$

$$c = \frac{1}{2}\left(\frac{1+(x\bar{y})_{\mathrm{S}}}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} - 1\right) \tag{35}$$

and we think of vector quaternions in (31) as of simple 3-vectors and round and squared brackets denote the usual operations of vector algebra: scalar and vector products, respectively.

3. Projector representation and curvatures

3.1. Hamiltonian in terms of projector-valued operators

The technique involving projector-valued operators is a powerful tool in gauge fields theory on the whole [14]. The remarkable properties of the system in question, namely the appearance of the projective operators in the Hamiltonian (22), encourage one to study its structure in more detail. It turns out [15, 16] that $H_0(x)$ (23) can be presented in terms of projectors:

$$H_0(x) = P_1(x) - P_2(x) \tag{36}$$

$$P_1(x) = \frac{1}{1+|x|^2}\begin{pmatrix} 1 & x \\ \bar{x} & |x|^2 \end{pmatrix} \qquad P_2(x) = \frac{1}{1+|x|^2}\begin{pmatrix} |x|^2 & -x \\ -\bar{x} & 1 \end{pmatrix}. \tag{37}$$

So we obtain the expression for the Hamiltonian (22) via projective operators

$$\begin{aligned} H &= P_+(P_1(x) - P_2(x)) + P_-(P_1(y) - P_2(y)) \\ &= [P_+P_1(x) + P_-P_1(y)] - [P_+P_2(x) + P_-P_2(y)]. \end{aligned} \tag{38}$$

We introduce

$$\Pi_+ = P_+P_1(x) + P_-P_1(y) \tag{39}$$

$$\Pi_- = P_+P_2(x) + P_-P_2(y) \tag{40}$$

where $\Pi_\pm$ project onto the orthogonal subspaces corresponding to eigenvalues ± 1, respectively. So $\Pi_+\Pi_- = \Pi_-\Pi_+ = 0$. Notice also that $P_\pm$ commute with $P_q(z)$, $q = 1, 2$, $z = x, y$.

3.2. Curvatures

The above representation appears to be very appropriate for the description of the geometry of the problem. We start from the left-invariant Maurer–Cartan form

$$\Theta = U^\dagger\, \mathrm{d}U = \begin{pmatrix} A_+ & -L \\ L^\dagger & A_- \end{pmatrix} \tag{41}$$

which satisfies the equation

$$\mathrm{d}\Theta + \Theta \wedge \Theta = 0. \tag{42}$$

We find that [17]

$$\begin{aligned} L \wedge L^\dagger &= \mathrm{d}A_+ + A_+ \wedge A_+ \equiv F_+ \\ -\mathrm{d}L &= A_+ \wedge L + L \wedge A_- \\ L^\dagger \wedge L &= \mathrm{d}A_- + A_- \wedge A_- \equiv F_-. \end{aligned} \tag{43}$$

Let us define the following forms:

$$J \overset{\text{def}}{=} \tfrac{1}{2}[\Theta, \gamma_5] = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \tag{44}$$

$$\Omega \overset{\text{def}}{=} J \wedge J = \begin{pmatrix} L \wedge L^\dagger & 0 \\ 0 & L^\dagger \wedge L \end{pmatrix} = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}. \tag{45}$$

Notice that

$$\begin{aligned} \mathrm{d}\Pi_\sigma &= \mathrm{d}\tfrac{1}{2}(1 + \sigma H) = \tfrac{1}{2}\sigma\left(\mathrm{d}U\,\gamma_5 U^\dagger + U\gamma_5\,\mathrm{d}U^\dagger\right) \\ &= \tfrac{1}{2}\sigma U(\Theta\gamma_5 - \gamma_5\Theta)U^\dagger = \sigma U J U^\dagger \end{aligned} \tag{46}$$

where $\sigma = +$ or $-$. Then

$$\mathrm{d}\Pi_\sigma \wedge \mathrm{d}\Pi_\sigma = U(J \wedge J)U^\dagger \tag{47}$$

and

$$\Omega = U^\dagger(\mathrm{d}\Pi_\sigma \wedge \mathrm{d}\Pi_\sigma)U. \tag{48}$$

Finally, we obtain the expression for curvatures

$$F_\sigma = \mathrm{Tr}_{\mathbb{H}}\left(\tfrac{1}{2}(1 + \sigma\gamma_5)\Omega\tfrac{1}{2}(1 + \sigma\gamma_5)\right). \tag{49}$$

The trace is assumed to be taken over 2×2 matrices defined over the body of quaternions $\mathbb{H}$. Noticing that

$$\Pi_\sigma = \tfrac{1}{2}(1 + \sigma H) = U\tfrac{1}{2}(1 + \sigma\gamma_5)U^\dagger \tag{50}$$

we come to the well known formula [9]

$$F_\sigma = \mathrm{Tr}_{\mathbb{H}}\left(U^\dagger\Pi_\sigma(\mathrm{d}\Pi_\sigma \wedge \mathrm{d}\Pi_\sigma)\Pi_\sigma U\right). \tag{51}$$

With $F_\pm$ being fully determined by J, we quote the expression for it:

$$\begin{aligned} J &= \left(\sqrt{\alpha_+}P_+\,\mathrm{d}X + \sqrt{\alpha_-}P_-\,\mathrm{d}Y\right)\left(\sqrt{\alpha_+}P_+ + \sqrt{\alpha_-}P_-\right) + \mathrm{d}P_+\,\beta(X - Y) \\ &= \alpha_+ P_+\,\mathrm{d}X\,P_+ + \alpha_-\,P_-\,\mathrm{d}Y\,P_- + \beta(X\,\mathrm{d}P_+\,P_- + P_-\,\mathrm{d}P_+\,X) + \beta(Y\,\mathrm{d}P_-\,P_+ + P_+\,\mathrm{d}P_-\,Y) \end{aligned} \tag{52}$$

where

$$X = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} = \gamma_a x_a \qquad Y = \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix} = \gamma_a y_a \tag{53}$$

$$\alpha_+ = \frac{1}{1 + |x|^2} \qquad \alpha_- = \frac{1}{1 + |y|^2} \qquad \beta = \sqrt{\alpha_+\alpha_-}. \tag{54}$$

The corresponding 2-form Ω can also be decomposed into $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ parts:

$$\Omega = \boldsymbol{\Omega} - \mathrm{i}\Omega_0. \tag{55}$$

4. Particular case: $SU(2)$ Yang–Mills instanton over S^4

If we put $x = y$ in (22) and (26) then

$$H = H_0(x) \tag{56}$$

and

$$U = U_0(x). \tag{57}$$

Pure quaternion structure arises (there is no explicit i in the expression) and the corresponding Hermitian bundle becomes equivalent to that associated with a quaternionic

Hopf bundle $\pi : S^7 \to S^4$, S^4 being the base and $SU(2) = Sp(1)$ being the structure group. As one can expect the $SU(2)$ Yang–Mills (anti)instantonic potential appears as the connection in this bundle:

$$\boldsymbol{A}_+ = \left(U^\dagger \, \mathrm{d}U\right)_{11} = \frac{(x \, \mathrm{d}\bar{x})_{\mathrm{V}}}{1 + |x|^2} \tag{58}$$

$$\boldsymbol{A}_- = \left(U^\dagger \, \mathrm{d}U\right)_{22} = \frac{(\bar{x} \, \mathrm{d}x)_{\mathrm{V}}}{1 + |x|^2} \tag{59}$$

($(A_0)_+ = (A_0)_- = 0$ as it must be).

Equation (52) takes the form

$$J = \frac{1}{1 + |x|^2} \, \mathrm{d}X \tag{60}$$

which allows one to calculate readily the curvatures of instantonic and anti-instantonic potentials

$$\Omega = \frac{1}{\left(1 + |x|^2\right)^2} \begin{pmatrix} \mathrm{d}x \wedge \mathrm{d}\bar{x} & 0 \\ 0 & \mathrm{d}\bar{x} \wedge \mathrm{d}x \end{pmatrix} = \begin{pmatrix} F_{\mathrm{inst}} & 0 \\ 0 & F_{\text{anti-inst}} \end{pmatrix}. \tag{61}$$

The geometric phase for the Hamiltonian (56) has been studied in [9, 10, 16]. It takes place in certain physical models, for instance, in time-reversal invariant systems of fermions (adiabatic evolution of two Kramers doublets [7]), systems with quadrupole interaction [8]. (For more details see [9].)

5. Particular case: Hermitian bundle over $\mathbb{CP}^2$

Since the complex projective space $\mathbb{CP}^2$ is the submanifold of the complex Grassmannian manifold $\mathbb{CG}(4, 2)$, a mathematical description of the Hermitian bundle over $\mathbb{CG}(4, 2)$ contains the description of that over $\mathbb{CP}^2$ as a particular case. It should be understood that physically these problems are essentially different, the particular is meant in a mathematical sense only. The latter bundle corresponds to the cyclic evolution of the system with one twice-degenerate level and another non-degenerate one. This problem has been studied recently [11–13] and the expressions for both Abelian ($\mathfrak{u}(1)$) and non-Abelian ($\mathfrak{su}(2)$) parts of the connection have been obtained. We shall demonstrate how to recover them in our framework and discuss some features of the connections concerned which become transparent due to the chosen parametrization.

It is obvious that the evolution operator for the system in question is presented as

$$U = \exp \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ -\xi_1^* & -\xi_2^* & 0 \end{pmatrix} \tag{62}$$

where $\xi_1 = a_1 + \mathrm{i}b_1$ and $\xi_2 = a_2 + \mathrm{i}b_2$ are arbitrary complex numbers.

If we add auxiliary row and column of zeros

$$U = \exp \begin{pmatrix} 0 & 0 & \xi_1 & 0 \\ 0 & 0 & \xi_2 & 0 \\ -\xi_1^* & -\xi_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{63}$$

then the evolution operator will have the following structure:

$$U = \begin{pmatrix} & 3\times 3 & & 0 \\ & \text{meaningful} & & 0 \\ & \text{block} & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

If we identify

$$2a_1 = u_3 = -v_4 \tag{65}$$

$$2b_1 = u_4 = v_3 \tag{66}$$

$$2a_2 = u_1 = -v_2 \tag{67}$$

$$2b_2 = u_2 = v_1 \tag{68}$$

then instead of (63) we can use (9) with the restriction on the parameters

$$(\mathrm{i}u - v)\tfrac{1}{2}(1 - \mathrm{i}e_3) = 0. \tag{69}$$

To make it clear, notice that projector $\frac{1}{2}(1 - \mathrm{i}e_3)$ in σ-matrix representation has the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The condition (9) leads to

$$v = ue_3 \tag{70}$$

$$\bar{u}v = |u|^2 e_3 \qquad v\bar{u} \equiv \tfrac{1}{2}\boldsymbol{r} = ue_3\bar{u} \qquad \hat{\boldsymbol{r}} = \frac{ue_3\bar{u}}{|u|^2} \tag{71}$$

and (9) takes the form

$$U = P_+ \cos 2|v| + P_- + P_+ \frac{\sin 2|v|}{2|v|} \begin{pmatrix} 0 & -2v \\ 2\bar{v} & 0 \end{pmatrix} \tag{72}$$

or, in parametrization (20) and (21),

$$U = P_+ \frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} 0 & -x \\ \bar{x} & 0 \end{pmatrix} + P_- = P_+ U_0(x) + P_- \tag{73}$$

with

$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 \pm \mathrm{i}\hat{\boldsymbol{r}} & 0 \\ 0 & 1 \pm \mathrm{i}e_3 \end{pmatrix} \qquad \hat{\boldsymbol{r}} = \hat{x}e_3\hat{\bar{x}}. \tag{74}$$

The initial Hamiltonian of the system is

$$H(0) = \tfrac{3}{2}\gamma_5 - \tfrac{1}{2} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & -2 \end{pmatrix} \tag{75}$$

and we recall that only left upper 3×3 block is meaningful for us. At arbitrary time

$$H(t) = \frac{1}{2(1+|x|^2)} \begin{pmatrix} 2 - |x|^2 - 3\mathrm{i}\hat{\boldsymbol{r}}|x|^2 & 3x(1+\mathrm{i}e_3) \\ 3(1+\mathrm{i}e_3)\bar{x} & 2|x|^2 - 1 - 3\mathrm{i}e_3 \end{pmatrix} \tag{76}$$

and passing to a σ-matrix representation one should also extract the necessary block.

From (31) and (32) we find non-Abelian

$$\boldsymbol{A} = \left(1 - \frac{1}{\sqrt{1+|x|^2}}\right)\hat{x}\,\mathrm{d}\hat{\bar{x}} - \hat{\boldsymbol{r}}\left(\frac{1}{2}\frac{|x|^2}{1+|x|^2} + \frac{1}{\sqrt{1+|x|^2}} - 1\right)(\hat{\bar{x}}\,\mathrm{d}\hat{x})_3 \tag{77}$$

and Abelian

$$A_0 = -\frac{1}{2}\frac{(\hat{\bar{x}}\,\mathrm{d}\hat{x})_3}{1+|x|^2}\,e_0 \tag{78}$$

parts of the connection. $(\hat{\bar{x}}\,\mathrm{d}\hat{x})_3$ denotes the third component of the vector quaternion $(\hat{\bar{x}}\,\mathrm{d}\hat{x})_{\mathrm{V}}$.

To make these expressions more convenient for further analysis, let us introduce two other parametrizations according to

$$\frac{1}{\sqrt{1+|x|^2}} \equiv \cos 2|v| = \frac{|z|^2-1}{|z|^2+1} = \frac{1-|w|^2}{1+|w|^2} \tag{79}$$

$$\frac{|x|}{\sqrt{1+|x|^2}} \equiv \sin 2|v| = \frac{2|z|}{|z|^2+1} = \frac{2|w|}{1+|w|^2} \tag{80}$$

$$|z| = \cot|v| \qquad 1 \leqslant |z| < \infty$$

$$|w| = 1/|z| = \tan|v| \qquad 0 < |w| \leqslant 1.$$

Then (77) and (78) become

$$\begin{aligned}
\boldsymbol{A} &= \frac{2}{1+|z|^2}\hat{z}\,\mathrm{d}\hat{\bar{z}} + \hat{\boldsymbol{r}}\frac{2}{\left(1+|z|^2\right)^2}(\hat{\bar{z}}\,\mathrm{d}\hat{z})_3 \\
&= \frac{2}{\left(1+|z|^2\right)|z|^2}(z\,\mathrm{d}\bar{z})_{\mathrm{V}} + \hat{\boldsymbol{r}}\frac{2}{\left(1+|z|^2\right)^2|z|^2}(\bar{z}\,\mathrm{d}z)_3 \\
&= \frac{2|w|^2}{1+|w|^2}\hat{w}\,\mathrm{d}\hat{\bar{w}} + \hat{\boldsymbol{r}}\frac{2|w|^4}{\left(1+|w|^2\right)^2}(\hat{\bar{w}}\,\mathrm{d}\hat{w})_3 \\
&= \frac{2}{1+|w|^2}(w\,\mathrm{d}\bar{w})_{\mathrm{V}} + \hat{\boldsymbol{r}}\frac{2|w|^2}{\left(1+|w|^2\right)^2}(\bar{w}\,\mathrm{d}w)_3
\end{aligned} \tag{81}$$

$$A_0 = -\frac{2(\bar{z}\,\mathrm{d}z)_3}{\left(1+|z|^2\right)^2} = -\frac{2(\bar{w}\,\mathrm{d}w)_3}{\left(1+|w|^2\right)^2} \tag{82}$$

where $\hat{z} = z/|z|$ and $\hat{w} = w/|w|$. It is evident that A_0 is the same in both parametrizations. If we perform the local gauge transformation

$$\boldsymbol{A}' = -\hat{\boldsymbol{r}}\boldsymbol{A}\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}\,\mathrm{d}\hat{\boldsymbol{r}} = \frac{2|z|^2}{1+|z|^2}\hat{z}\,\mathrm{d}\hat{\bar{z}} + \hat{\boldsymbol{r}}\frac{2|z|^4}{\left(1+|z|^2\right)^2}(\hat{\bar{z}}\,\mathrm{d}\hat{z})_3 \tag{83}$$

we will see that $\boldsymbol{A}(z) = \boldsymbol{A}'(w)$.

Let us perform another gauge transformation

$$\boldsymbol{A}'' = \hat{\bar{z}}\boldsymbol{A}\hat{z} + \hat{\bar{z}}\,\mathrm{d}\hat{z} = \frac{|z|^2-1}{|z|^2+1}\hat{\bar{z}}\,\mathrm{d}\hat{z} + \frac{2}{\left(|z|^2+1\right)^2}e_3(\hat{\bar{z}}\,\mathrm{d}\hat{z})_3. \tag{84}$$

Potential $\boldsymbol{A}''$ has the same functional dependence on z as globally equivalent to it

$$\boldsymbol{A}''' = -e_3\boldsymbol{A}''e_3 = \frac{|w|^2-1}{|w|^2+1}\,\hat{\bar{w}}\,\mathrm{d}\hat{w} + \frac{2}{\left(|w|^2+1\right)^2}e_3(\hat{\bar{w}}\,\mathrm{d}\hat{w})_3 \tag{85}$$

has on w. Notice that the transformation $|z| \to |w| = 1/|z|$ is not full inversion. To become the latter, it should be composed with transformation $\hat{z} \to \hat{\bar{z}}$.

In conclusion, we quote the expressions for curvatures (in the initial gauge). They are taken from (43)–(45) and (52) with the restriction (69) being valid:

$$L = \frac{1}{1+|x|^2}\frac{1+\mathrm{i}\hat{\boldsymbol{r}}}{2}\mathrm{d}x\frac{1+\mathrm{i}e_3}{2} \tag{86}$$

$$F_+ = L\wedge L^\dagger = \frac{1}{\left(1+|x|^2\right)^2}\frac{1+\mathrm{i}\hat{\boldsymbol{r}}}{2}\mathrm{d}x\frac{1+\mathrm{i}e_3}{2}\wedge\mathrm{d}\bar{x}\frac{1+\mathrm{i}\hat{\boldsymbol{r}}}{2}$$

$$= \frac{1}{|x|\left(1+|x|^2\right)^2}(\mathrm{i}-\hat{\boldsymbol{r}})\,\mathrm{d}|x|\wedge(\bar{x}\,\mathrm{d}x)_3 \tag{87}$$

$$F_- = L^\dagger\wedge L = \frac{1+\mathrm{i}e_3}{2}\left(-\frac{2\mathrm{i}}{|x|\left(1+|x|^2\right)^2}\,\mathrm{d}|x|\wedge(\bar{x}\,\mathrm{d}x)_3\right)\frac{1+\mathrm{i}e_3}{2}. \tag{88}$$

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